Alexander V. Osipov, Remarks on Ostrovsky's theorem, *Sib. Èlektron. Mat. Izv.*, 2019, Volume 16, 435–438

DOI: https://doi.org/10.33048/semi.2019.16.025

S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 435–438 (2019) УДК 515.126, 515.124, 515.128
DOI 10.33048/semi.2019.16.025 MSC 26A15, 54C08, 26A21, 54H05, 54E40

REMARKS ON OSTROVSKY'S THEOREM

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Abstract. In this paper we prove that the condition *'one-to-one'* of the continuous open-resolvable mapping is necessary in the Ostrovsky theorem (Theorem 1 in [4]). Also we get that the Ostrovsky problem ([6], Problem 2) (*Is every continuous open-LC_n function between Polish spaces piecewise open for $n = 2, 3, ...$?*) has a negative solution for each $n > 1$.

Keywords: open-resolvable function, open function, resolvable set, open-LC_n function, piecewise open function, scatteredly open function.

1. Introduction

By theorem of Jayne and Rogers a function $f : X \mapsto Y$ between Polish spaces is Δ^0_2-measurable iff it is piecewise continuous (see [2, 3]).

The generalization of theorem of Jayne and Rogers to multi-valued functions raises some topological problems for single-valued functions (see [6], Problem 1 and 2).

In the following definitions we will suppose that X is a subspace of the Cantor set $\mathbf{C}$.

A function $f : X \mapsto Y$ is called *piecewise open* if X admits a countable, closed and disjoint cover $\mathcal{V}$, such that for each $V \in \mathcal{V}$ the restriction $f|V$ is open.

Recall, that a subset E of a metric space X is *resolvable* [1], if for each nonempty closed in X subset F we have $cl_X(F \cap E) \cap cl_X(F \setminus E) \neq F$.

If $E \subset X$ is resolvable, then E is Δ^0_2-set in X and vice versa if the space X is Polish (= separable complete metrizable).

Recall that a subset of X is LC_n-set if it can be written as union of n locally closed in X sets (a set is locally closed if it is an intersection of an open and a closed set). Every LC_n-set (constructible) set is resolvable.

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Received October, 4 2018, published March, 29, 2019.

A mapping f is open if it maps open sets onto open ones. More generally, for $n \in \omega$ a mapping f is said to be *open-resolvable* (open-LC_n) if f maps open set onto resolvable (LC_n-set) ones.

A piecewise open function $f : X \mapsto Y$ is called *scatteredly open* if, in addition, the cover $\mathcal{V}$ is scattered, that is: for every nonempty subfamily $\mathcal{T} \subset \mathcal{V}$ there is a clopen set $G \subset X$ such that $\mathcal{T}_G = \{T \in \mathcal{T} : T \subset G\}$ is a singleton and $T \bigcap G = \emptyset$ for every $T \in \mathcal{T} \setminus \mathcal{T}_G$.

2. Main result

A.V. Ostrovsky proved the following result.

Theorem 1. *(Theorem 1 in* [4]*) Let X and Y be subspaces of the Cantor set $\mathbf{C}$, and $f : X \mapsto Y$ a continuous bijection. If the image under f of every open set in X is resolvable in Y, then f is scatteredly open, and, hence, f is scattered homeomorphism.*

Theorem 2. *(Proposition 3.2 in* [5]*) Every continuous open-LC_1 function $X \mapsto Y$ onto a metrizable crowded space Y is open.*

In ([6], Problem 2) A.V. Ostrovsky posed the following

Problem. Is every continuous open-LC_n function between Polish spaces piecewise open for $n = 2, 3, ...$?

We prove that
- the condition *'one-to-one'* of the mapping f in Theorem 1 is necessary.
- Ostrovsky's problem has a negative solution for $n = 2$ (hence for each $n > 1$).

Example. Let $\mathbf{C}$ be the Cantor set such that $\mathbf{C} \subset [0, 1]$. As usually, we starts be deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0, 1]$, leaving two segments: $P_1 = C_0 \cup C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four segments: $P_2 = C_{00} \cup C_{02} \cup C_{20} \cup C_{22} = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. This process is continued ad infinitum, where the nth set is $P_n = \frac{P_{n-1}}{3} \cup (\frac{2}{3} + \frac{P_{n-1}}{3})$ for $n \geq 1$, and $P_0 = [0, 1]$.

The Cantor ternary set contains all points in the interval $[0, 1]$ that are not deleted at any step in this infinite process:

$\mathbf{C} := \bigcap\limits_{n=1}^{\infty} P_n$.

Consider the clopen base $\mathcal{B} := \{\mathbf{C} \cap C_{s_1,...,s_k} : s_i \in \{0, 2\}, i \in \overline{1,k},\, k \in \omega\}$ on $\mathbf{C}$. We enumerate $\mathcal{B} = \{B_n : n \in \omega\}$ as $B_1 = C_0 \cap \mathbf{C}$, $B_2 = C_2 \cap \mathbf{C}$, $B_3 = C_{00} \cap \mathbf{C}$, $B_4 = C_{02} \cap \mathbf{C}$, ..., $B_n = C_{s(n)} \cap \mathbf{C}$, ..., where $s(n)$ is the binary representation of the number $n + 1$ without the first digit and the digit 1 must be replaced by 2. Consider the countable dense set $\{b_n : n \in \omega\}$ such that $b_n \in B_n$, $b_n \neq b_m$ $(n \neq m)$ for $n, m \in \omega$.

Let us fix the countable dense set $\{(a_n, b_n) : n \in \omega\}$ in $\mathbf{C} \times \mathbf{C}$ such that $a_n \neq a_m$ for $n \neq m$, and for each n pick $a_{n,i} \mapsto a_n$ such that $a_{n,i} \neq a_m$, $a_{n,i} \neq a_{m,j}$ for $(n, i) \neq (m, j)$, and $|a_{n,i} - a_n| < \frac{1}{n}$.

Let $X = (\mathbf{C} \times \mathbf{C}) \setminus \bigcup\limits_{n,i} \{a_{n,i}\} \times B_n$. Note that X is a G_δ-set in $\mathbf{C} \times \mathbf{C}$. It follows that X is a Polish space.

Let $\pi|X : X \mapsto \mathbf{C}$ be a restriction to X of the projection $\pi : \mathbf{C} \times \mathbf{C} \mapsto \mathbf{C}$ onto the first coordinate. Note that $\pi(X) = \mathbf{C}$ because of $diam\mathbf{C} > diamB_n$ for any $n \in \omega$.

Suppose $X = \bigcup\limits_{n\in\omega} X_n$ is a countable union of closed subsets X_n of X. By the Baire Category Theorem [1], there is X_m such that $V = IntX_m \neq \emptyset$ because otherwise $\bigcap\limits_{n=1}^{\infty} (X \setminus X_n)$ is not a dense set in X.

Since the set $\{(a_n, b_n) : n \in \omega\}$ is dense in X, there are $n' \in \omega$ and $W \in \mathcal{B}$ such that the point $(a_{n'}, b_{n'}) \in ((W \times B_{n'}) \cap X) \subset V$. Since the set $\{(a_n, b_n) : n \in \omega\}$ is dense in $(W \times B_{n'}) \cap X$, choose $n'' \in \omega$ such that $n'' > n'$ and $(a_{n''}, b_{n''}) \in ((W \times B_{n''}) \cap X) \subset (W \times B_{n'}) \cap X$. Then $\pi|X_m : X_m \mapsto \pi(X_m)$ is not open at $(a_{n''}, b_{n''})$ because of $\pi((W \times B_{n''}) \cap X)$ does not contain $\{a_{n'',i} : i \in \omega\}$ and hence it is not an open subset of $\pi(X_m)$. Therefore $\pi|X$ is not piecewise open and hence it is not scatteredly open.

Let $U \subset \mathbf{C} \times \mathbf{C}$ be open. We have to check that $\pi(U \cap X) \in \Delta^0_2$.

Construct for every point $(a, b) \in U \cap X$ sets $W(a)$ and $B(b)$ such that

• $a \in W(a) \in \mathcal{B}$, $b \in B(b) \in \mathcal{B}$ and $(W(a) \times B(b)) \cap X \subset U$.

• if $a \neq a_m$ for any $m \in \omega$, then $\pi((W(a) \times B(b)) \bigcap X) = W(a)$.

• if $a = a_m$ for some $m \in \omega$, then $\pi((W(a) \times B(b)) \bigcap X) = W(a) \setminus \{a_{m,i_j} : j \in \omega\}$ for some subsequence $\{a_{m,i_j} : j \in \omega\} \subseteq \{a_{m,i} : i \in \omega\}$.

Case 1. Let $a \neq a_m$ and $a \neq a_{m,i}$ for any $m, i \in \omega$. One can choose W, $B(b) = B_{n'} \in \mathcal{B}$ such that $a \in W$, $b \in B(b)$, $(W \times B(b)) \cap X \subset U$. Since $a \neq a_m$ for any $m \in \omega$, then there exists $W(a) \in \mathcal{B}$ such that $a \in W(a) \subset W$ and $W(a) \cap \{a_i \cup \{a_{i,j} : j \in \omega\} : i \in \overline{1, n'}\} = \emptyset$. Then $\pi((W(a) \times B(b)) \bigcap X) = W(a)$.

Case 2. Let $a = a_{m,i}$ for some $m, i \in \omega$. One can choose W, $B(b) = B_{n'} \in \mathcal{B}$ such that $a \in W$, $b \in B(b)$, $(W \times B(b)) \cap X \subset U$.

If $m > n'$ then there exists $W(a) \in \mathcal{B}$ such that $a \in W(a) \subset W$ and $W(a) \cap \{a_i \cup \{a_{i,j} : j \in \omega\} : i \in \overline{1, n'}\} = \emptyset$. Then $\pi((W(a) \times B(b)) \bigcap X) = W(a)$.

If $m \leq n'$ then there exists $W(a) \in \mathcal{B}$ such that $a \in W(a) \subset W$ and $W(a) \cap (\{a_i \cup \{a_{i,j} : j \in \omega\} : i \in \overline{1, n'}\}) \setminus \{a_{m,i}\} = \emptyset$. Then $\pi((W(a) \times B(b)) \bigcap X) = W(a)$, too.

Case 3. Let $a = a_m$ for some $m \in \omega$. Analogously to Case 1, we can choose $B(b) \in \mathcal{B}$ such that $B(b) \setminus B_n \neq \emptyset$ for all $n > n' > m$, and $W(a) \in \mathcal{B}$ can choose such that $W(a) \cap \{a_i \cup \{a_{i,j} : j \in \omega\} : i \in \overline{1, n'}$ and $i \neq m\} = \emptyset$.

Then $W(a) \setminus \pi((W(a) \times B(b)) \bigcap X) \subset \{a_{m,i} : i \in \omega\}$ and hence $\pi((W(a) \times B(b)) \bigcap X) = W(a) \setminus \{a_{m,i_j} : j \in \omega\} = W_a \cup \{a_m\}$ where $W_a = W(a) \setminus (\{a_m\} \cup \{a_{m,i_j} : j \in \omega\})$ is open in $\mathbf{C}$.

Thus

$$\begin{aligned}\pi(U \cap X) &= \bigcup_{(a,b)\in U\cap X} \pi((W(a) \times B(b)) \bigcap X) \\ &= \Big(\bigcup_{(a,b)\in U\cap X, a\neq a_m} W(a)\Big) \cup \Big(\bigcup_{(a,b)\in U\cap X, a=a_m} W_a \cup \{a_m\}\Big).\end{aligned}$$

By definition of the clopen base $\mathcal{B}$, $\pi(U \cap X) = S \cup D$ where

$$S = \Big(\bigcup_{(a,b)\in U\cap X, a\neq a_m} W(a)\Big) \cup \Big(\bigcup_{(a,b)\in U\cap X, a=a_m} W_a\Big)$$

is an open set in $\mathbf{C}$ and $D = \{a_{m_k} : k \in \omega\}$ is a discrete in itself set such that $S \bigcap D = \emptyset$. Indeed, by Case 3, for every $a_{m_k} \in D$ there is $W(a_{m_k}) \in \mathcal{B}$ such that $a_{m_k} \in W(a_{m_k})$ and $W(a_{m_k}) \bigcap \{a_{m_i} : i \in \omega, i \neq k\} = \emptyset$. It follows that D is discrete in itself. Hence $\pi(U \cap X)$ is Δ^0_2. Since $\pi(X) = \mathbf{C}$ is Polish, the mapping $\pi|X$ is continuous open-resolvable.

Note that $\pi(U \cap X) = S \cup ((\bigcup\limits_{(a,b) \in U \cap X} W(a)) \bigcap \overline{D})$. It follows that $\pi(U \cap X)$ is LC_2-set and hence $\pi|X$ is open-LC_2.

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